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Fractional variational principles with delay

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Abstract

The fractional variational principles within Riemann–Liouville fractional derivatives in the presence of delay are analyzed. The corresponding Euler–Lagrange equations are obtained and one example is analyzed in detail.

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1. Introduction

The calculus of variations has communication with some branches of sciences and engineering, e.g. differential equations, geometry, control theory, economics, electrical engineering and so on [1–8]. Very recently, Heymans and Podlubny [8] analyzing a series of examples from the field of viscoelasticity have proved that it is possible to assign a physical interpretation meaning by using the Riemann–Liouville fractional derivatives.

For the last three decades, a considerable amount of research has advanced our understanding of the effect of time delays on the behavior of a dynamic system. These delays, which may either exist within the system's internal states or are introduced through a closed-loop feedback, produce complex dynamic responses [23–26]. The combined use of fractional derivatives and delay [23] was investigated for the stability analysis of linear fractional-differential system with multiple time scales [24].

In very recent years, the fractional variational principles have been developed and applied to the control problems or physical problems [9–22]. The investigations on these issues are based to a large degree on the replacement of the classical derivatives by the fractional ones, especially Riemann–Liouville and Caputo fractional derivatives. Another issue is to apply the fractional calculus properties for the fractional derivatives and fractional integrals, which are the generalizations of the classical ones to any order α, and to obtain a series of new fractional Euler–Lagrange, Hamilton and Hamilton–Jacobi equations.

However, it is found that the replacement of the classical derivatives with the fractional ones produces a theory which differs from the classical one except for the limit when $\alpha \to 1$. By following the above prescription, for a given fractional Lagrangian, non-locality is

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introduced naturally in fractional derivatives but not in the potential part. Therefore, to quantize properly the fractional theories we have to introduce the non-locality into the potential part too. This idea leads naturally to a Lagrangian theory with both fractional derivatives and delay. To the best of our knowledge the variational principles combined with fractional derivatives and delay have not been investigated yet in the literature. Despite several methods proposed for the fractional quantization issue, none of them considered the implication of the delay and fractional derivatives.

The main aim of this paper is to find the appropriate Euler–Lagrange equation for a fractional Lagrangian possessing delay terms.

2. Basic definitions

In this section we mention some basic definitions and results which we use in the following sections.

The left Riemann–Liouville fractional derivative is defined by

$$ {}_{t_1}\mathbf{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t_1}^{t} (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{1} $$

The right Riemann–Liouville fractional derivative is defined by

$$ {}_{t}\mathbf{D}_{t_2}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t}^{t_2} (\tau-t)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad n = [Re(\alpha)] - 1. \tag{2} $$

The fractional derivative of a constant, interestingly, takes the form

$$ {}_{t_1}\mathbf{D}_t^{\alpha} C = C\frac{(t-t_1)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{3} $$

and the fractional derivative of a power of t has the following form,

$$ {}_{t_1}\mathbf{D}_t^{\alpha} (t-t_1)^{\beta} = \frac{\Gamma(\beta+1)(t-t_1)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, \tag{4} $$

for $\beta > -1, \alpha \geqslant 0$.

The composite of fractional derivatives has the following form,

$$ {}_{t_1}\mathbf{D}_t^{\alpha}\, {}_{t_1}\mathbf{D}_t^{\sigma} f(t) = {}_{t_1}\mathbf{D}_t^{\alpha+\sigma} f(t) - \sum_{j=1}^{k} {}_{t_1}\mathbf{D}_t^{\sigma-j} f(t)|_{t=t_1} \frac{(t-t_1)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \tag{5} $$

where $0 \leqslant k-1 < \sigma \leqslant k, n-1 < \alpha \leqslant n, \alpha > 0, \sigma > 0, \alpha+\sigma < n$ and k is a whole number. The fractional product rule has the form

$$ {}_{t_1}\mathbf{D}_t^{\alpha}(fg) = \sum_{j=0}^{\infty} \binom{\alpha}{j} \left({}_{t_1}\mathbf{D}_t^{\alpha-j} f\right) \left(\frac{\mathrm{d}^j g}{\mathrm{d}t^j}\right). \tag{6} $$

For $\alpha > 0$, $1 \leqslant p \leqslant \infty$, the function spaces ${}_{t_1}I^{\alpha}(L_p)$ and $I_{t_2}^{\alpha}(L_p)$ are defined by

$$ {}_{t_1}I^{\alpha}(L_p) = \{f : f = {}_{t_1}I^{\alpha}\varphi, \varphi \in L_p(t_1, t_2)\} \tag{7} $$

and

$$ I_{t_2}^{\alpha}(L_p) = \{f : f = I_{t_2}^{\alpha}\phi, \phi \in L_p(t_1, t_2)\} \tag{8} $$

respectively. Here, ${}_{t_1}I^{\alpha}$ and $I_{t_2}^{\alpha}$ are the left and right Riemann–Liouville functional integrals, respectively.

The following lemma presents the rules for fractional integration by parts over the whole interval $T = (t_1, t_2)$.

Lemma 1. *[1] Let $\alpha > 0$, $p, q \geqslant 1$ and $\frac{1}{p} + \frac{1}{q} \leqslant 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case where $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$)*

(a) If $\varphi \in L_p(t_1, t_2)$ and $\psi \in L_q(t_1, t_2)$, then

$$\int_{t_1}^{t_2} \varphi(t)({}_{t_1}I^\alpha \psi)(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \psi(t)\big(I_{t_2}^\alpha \varphi\big)(t)\,\mathrm{d}t. \tag{9}$$

(b) If $g \in I_{t_2}^\alpha(L_p)$ and $f \in_{t_1} I^\alpha(L_q)$, then

$$\int_{t_1}^{t_2} g(t)\big({}_{t_1}\mathbf{D}_t^\alpha f\big)(t)\,\mathrm{d}t = \int_{t_1}^{t_2} f(t)\big({}_t\mathbf{D}_{t_2}^\alpha g\big)(t)\,\mathrm{d}t. \tag{10}$$

The above properties of the fractional derivatives make them suitable to describe complex systems. The non-local effects encoded in the definition of the fractional derivatives are suitable for describing the phenomena possessing a memory effect. A delay differential equation represents a special type of functional differential equation (for more details see [23]). As is known the delay differential equations are similar to ordinary differential equations; however their evolution implies the past values of the state variable. As a result, the solution of delay differential equations requires knowledge of not only the current state, but also of the state of a previously given time. In addition to that the fractional delay variational problems may arise in connection with nonconservative systems.

3. Fractional Euler–Lagrange equations with delay

First we state a lemma, which is an improvement on lemma 1, that will serve in proving the main result in theorem 1.

Lemma 2. *Let $\alpha > 0$, $p, q \geqslant 1$, $r \in T = (t_1, t_2)$ and $\frac{1}{p} + \frac{1}{q} \leqslant 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case where $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).*

(a) If $\varphi \in L_p(t_1, t_2)$ and $\psi \in L_q(t_1, t_2)$, then

$$\int_{t_1}^{r} \varphi(t)({}_{t_1}I^\alpha \psi)(t)\,\mathrm{d}t = \int_{t_1}^{r} \psi(t)\big(I_r^\alpha \varphi\big)(t) \tag{11}$$

and hence,
if $g \in I_{t_2}^\alpha(L_p)$ and $f \in_{t_1} I^\alpha(L_q)$,
then

$$\int_{t_1}^{r} g(t)\big({}_{t_1}\mathbf{D}_t^\alpha f\big)(t)\,\mathrm{d}t = \int_{t_1}^{r} f(t)\big({}_t\mathbf{D}_r^\alpha g\big)(t)\,\mathrm{d}t. \tag{12}$$

(b) If $\varphi \in L_p(t_1, t_2)$ and $\psi \in L_q(t_1, t_2)$, then

$$\int_{r}^{t_2} \varphi(t)({}_{t_1}I^\alpha \psi)(t)\,\mathrm{d}t = \int_{r}^{t_2} \psi(t)\big(I_{t_2}^\alpha \varphi\big)(t)\,\mathrm{d}t + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{r} \psi(t)\left(\int_r^{t_2} \varphi(s)(s-t)^{\alpha-1}\,\mathrm{d}s\right)\mathrm{d}t$$

and hence,

if $g \in I_{t_2}^\alpha(L_p)$ and $f \in_{t_1} I^\alpha(L_q)$, then

$$\int_{r}^{t_2} g(t)\big({}_{t_1}\mathbf{D}_t^\alpha f\big)(t)\,\mathrm{d}t = \int_{r}^{t_2} f(t)\big({}_t\mathbf{D}_{t_2}^\alpha g\big)(t)\,\mathrm{d}t - \frac{1}{\Gamma(\alpha)}\int_{t_1}^{r} \big({}_{t_1}\mathbf{D}_t^\alpha f\big)(t)\left(\int_r^{t_2} \big({}_t\mathbf{D}_{t_2}^\alpha g\big)(s)(s-t)^{\alpha-1}\,\mathrm{d}s\right)\mathrm{d}t$$

which implies

$$\int_r^{t_2} g(t)\big({}_{t_1}\mathbf{D}_t^{\alpha} f\big)(t)\,\mathrm{d}t = \int_r^{t_2} f(t)\big({}_t\mathbf{D}_{t_2}^{\alpha} g\big)(t)\,\mathrm{d}t$$
$$- \frac{1}{\Gamma(\alpha)}\int_{t_1}^{r} f(t)_t\mathbf{D}_r^{\alpha}\left(\int_r^{t_2}\big({}_t\mathbf{D}_{t_2}^{\alpha} g\big)(s)(s-t)^{\alpha-1}\,\mathrm{d}s\right)\mathrm{d}t.$$

The proof is straightforward, by using theorem 3 of [27] to interchange the order of the integrals. For the fractional derivative case the result follows by using the definition of the function spaces ${}_{t_1}I^{\alpha}(L_q)$ and $I_{t_2}^{\alpha}(L_p)$.

In this section, we consider a modified problem when the fractional derivatives and delay both appear in the Lagrangian. More exactly, we consider the following one-dimensional problem:

minimize

$$J(y) = \int_{t_1}^{t_2} F(t, y(t), {}_{t_1}\mathbf{D}_t^{\alpha} y(t), y(t-\tau), y'(t-\tau))\,\mathrm{d}t, \tag{13}$$

such that

$$y(t_2) = b, \quad y(t) = \phi(t), \quad t \in [t_1-\tau, t_1], \quad t_1 < t_2, \quad \tau > 0, \quad \tau < t_2 - t_1. \tag{14}$$

By using the corresponding delay notations [23], namely

$$y_\tau = y(t-\tau), \qquad y'_\tau = y'(t-\tau) \tag{15}$$

equation (13) becomes

$$J(y) = \int_{t_1}^{t_2} F(t, y(t), {}_{t_1}\mathbf{D}_t^{\alpha} y(t), y_\tau, y'_\tau)\,\mathrm{d}t. \tag{16}$$

The next step is to define the following directional derivative of *J*(*y*) at *y*(*t*,*a*) in the direction of χ as follows:

$$J'(y,\chi) = \int_{t_1}^{t_2}\left[F_y\chi(t) + \frac{\partial F}{\partial\big({}_{t_1}\mathbf{D}_t^{\alpha} y(t)\big)}{}_{t_1}\mathbf{D}_t^{\alpha}\chi(t) + F_{y_\tau}\chi_\tau + F_{y'_\tau}\chi'_\tau\right]\mathrm{d}t. \tag{17}$$

If $y_0(t)$ represents a solution of the previous variational problem we define the variation of y with respect to a by $\chi(t) = \frac{\partial y(t,0)}{\partial a}$, where $y(t,a)$ denotes an admissible family obeying $y(t,0) = y_0(t)$ and $a \in R$ such that $0 <| a |< \epsilon$.

The Taylor series expansion for $y \in y(t,a)$ and J is

$$\begin{aligned} &y(t,a) = y_0(t) + a\chi(t) + O(a^2), t \in [t_1, t_2],\\ &J(y(t,a)) = J(y_0(t)) + aJ'(y(t,0), \chi(t)) + O(a^2). \end{aligned} \tag{18}$$

As a result we obtain

$$\begin{aligned} J'(y,\chi) &= \int_{t_1}^{t_2}\left[F_y\chi(t) + \frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}_t^{\alpha} y(t)\big)}{}_{t_1}\mathbf{D}_t^{\alpha}\chi(t) + F_{y_\tau}\chi_\tau + F_{y'_\tau}\chi'_\tau\right]\mathrm{d}t\\ &= \int_{t_1}^{t_2-\tau}\left[(F_y(t) + F_{y_\tau}(t+\tau))\chi(t) + \frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}_t^{\alpha} y(t)\big)}{}_{t_1}\mathbf{D}_t^{\alpha}\chi(t) + F_{y'_\tau}(t+\tau)\chi'(t)\right]\mathrm{d}t\\ &\quad + \int_{t_2-\tau}^{t_2}\left(F_y(t)\chi(t) + \frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}_t^{\alpha} y(t)\big)}{}_{t_1}\mathbf{D}_t^{\alpha}\chi(t)\right)\mathrm{d}t. \end{aligned} \tag{19}$$

In the above, we have made a change of variables for $t-\tau$ and have used the fact that $\chi = 0$ on $[t_1-\tau, t_1]$.

Now, if we apply the usual integration by parts and the fractional version stated in lemma 2 with $f = \chi$, we obtain

$$J'(y,\chi) = \int_{t_1}^{t_2-\tau}\Bigg[F_y(t) + F_{y_\tau}(t+\tau) + {}_t\mathbf{D}^\alpha_{t_2-\tau}\frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}^\alpha_t y(t)\big)}$$
$$-\frac{1}{\Gamma(\alpha)}{}_t\mathbf{D}^\alpha_{t_2-\tau}\int_{t_2-\tau}^{t_2}\left[{}_t\mathbf{D}^\alpha_{t_2}\left(\frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}^\alpha_t y(t)\big)}\right)\right](z)(z-t)^{\alpha-1}\,\mathrm{d}z$$
$$-\frac{\mathrm{d}F_{y'_\tau}(t+\tau)}{\mathrm{d}t}\Bigg]\chi(t)\,\mathrm{d}t + \int_{t_2-\tau}^{t_2}\left(F_y(t) +_t \mathbf{D}^\alpha_{t_2}\left(\frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}^\alpha_t y(t)\big)}\right)\right)\chi(t)\,\mathrm{d}t$$
$$+\big(F_{y'_\tau}(t+\tau)\big)\chi(t)\big|_{t_1}^{(t_2-\tau)^-} = 0. \tag{20}$$

Therefore, by taking into account (20) we obtain the following theorem:

Theorem 1. *Let J(y) be a functional of the form*

$$J(y) = \int_{t_1}^{t_2} F(t, y(t), {}_{t_1}\mathbf{D}^\alpha_t y(t), y(t-\tau), y'(t-\tau))\,\mathrm{d}t, \tag{21}$$

defined on a set of continuous functions y(t) which have continuous left fractional derivative of order α *in* $[t_1, t_2]$ *and satisfy the conditions* $y(t) = \phi(t), t \in [t_1-\tau, t_1]$ *and* $y(t_2) = a_2$. *Also let* $F : [t_1, t_2] \times R^4 \to R$ *have continuous partial derivatives with respect to all of its parameters and* $\phi(t)$ *be smooth. Then the necessary condition for J(y) to possess an extremum for a given function y(t) is that y(t) fulfils the following Euler–Lagrange equations:*

$$F_y(t) + F_{y_\tau}(t+\tau) + {}_t\mathbf{D}^\alpha_{t_2-\tau}\frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}^\alpha_t y(t)\big)} - \frac{\mathrm{d}F_{y'_\tau}(t+\tau)}{\mathrm{d}t}$$
$$-\frac{1}{\Gamma(\alpha)}{}_t\mathbf{D}^\alpha_{t_2-\tau}\int_{t_2-\tau}^{t_2}\left[{}_t\mathbf{D}^\alpha_{t_2}\left(\frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}^\alpha_t y(t)\big)}\right)\right](z)(z-t)^{\alpha-1}\,\mathrm{d}z = 0, \tag{22}$$

for $t_1 \leqslant t \leqslant t_2 - \tau$,

$$F_y(t) +_t \mathbf{D}^\alpha_{t_2}\left(\frac{\partial F(t)}{\partial\big({}_{t_1}\mathbf{D}^\alpha_t y(t)\big)}\right) = 0, \tag{23}$$

for $t_2 - \tau \leqslant t \leqslant t_2$, *as well as the boundary condition*

$$F_{y'_\tau}(t+\tau)\chi(t)\big|_{t_1}^{(t_2-\tau)^-} = 0. \tag{24}$$

It is observed from (22), (23) and (24) that when the delay terms are removed and in the limit $\alpha \to 1$, the classical results are reobtained.

3.1. Generalization

In several physical applications or in control theory the variational principles are very useful. The generalization of the previous theorem to the case of fixed end points and several functions is discussed below.

We assume that the functional $J(y_1, y_2, \ldots, y_n)$ is given by

$$J(y_1, y_2, \ldots, y_n) = \int_{t_1}^{t_2} F(t, y_1(t), \ldots y_n(t), {}_{t_1}\mathbf{D}^\alpha_t y_1(t) \ldots, {}_{t_1}\mathbf{D}^\alpha_t y_n(t),$$
$$y_1(t-\tau), \ldots, y_n(t-\tau), y'_1(t-\tau), \ldots, y'_n(t-\tau))\,\mathrm{d}t, \tag{25}$$

and it satisfies the following boundary conditions

$$y_i(t_2) = y_{it_2}, \qquad y_i(b) = y_{ib}, \qquad y_i(t) = \phi_i(t),$$
$$i = 1, \ldots, n, \quad t \in [t_1 - \tau, t_1], \quad t_1 < t_2, \quad \tau > 0, \quad \tau < t_2 - t_1. \tag{26}$$

Theorem 2. *Assume that F has continuous partial derivatives with respect to all of its parameters and ϕ_i, $i = 1, 2, \ldots, n$ are smooth. Then the necessary condition for the curves $y_i = y_i(t)$, $i = 1, \ldots, n$, fulfilling the conditions (26) to be an extremal of the functional (25) is that*

$$F_{y_i}(t) + F_{y_i\tau}(t+\tau) + {}_t\mathbf{D}^{\alpha}_{t_2-\tau} \frac{\partial F(t)}{\partial\left({}_{t_1}\mathbf{D}^{\alpha}_{t} y_i(t)\right)} - \frac{\mathrm{d}F_{y_i\tau'}(t+\tau)}{\mathrm{d}t}$$
$$- \frac{1}{\Gamma(\alpha)} {}_t\mathbf{D}^{\alpha}_{t_2-\tau} \int_{t_2-\tau}^{t_2} \left[{}_t\mathbf{D}^{\alpha}_{t_2} \left(\frac{\partial F(t)}{\partial\left({}_{t_1}\mathbf{D}^{\alpha}_{t} y_i(t)\right)} \right) \right](z)(z-t)^{\alpha-1}\,\mathrm{d}z = 0, \tag{27}$$

for $t_1 \leqslant t \leqslant t_2 - \tau$,

$$F_{y_i}(t) +_t \mathbf{D}^{\alpha}_{t_2} \left(\frac{\partial F(t)}{\partial\left({}_{t_1}\mathbf{D}^{\alpha}_{t} y_i(t)\right)} \right) = 0, \tag{28}$$

for $t_2 - \tau \leqslant t \leqslant t_2$ and the boundary conditions

$$\left(F_{y_i\tau'}\right)(t+\tau)\chi(t)\,\Big|_{t_1}^{(t_2-\tau)^-} = 0, \tag{29}$$

for $i = 1, \ldots, n$.

The proof can be done in the same manner as for theorem 1.

4. Example

In order to exemplify our results we analyze an example of physical interest. Namely, let us consider the following action,

$$J = \int_{t_1}^{t_2} \left[\frac{1}{2}\left({}_{t_1}\mathbf{D}^{\alpha}_{t} y(t)\right)^2 - V(y(t-\tau)) \right] \mathrm{d}t, \tag{30}$$

subject to the condition

$$y(t) = \phi(t), \qquad t \in [t_1 - \tau, t_1]. \tag{31}$$

The corresponding Euler–Lagrange equation is as follows,

$$-\frac{\partial V(t+\tau)}{\partial y_\tau} + {}_t\mathbf{D}^{\alpha}_{t_2-\tau}\, {}_{t_1}\mathbf{D}^{\alpha}_{t} y(t) - \frac{1}{\Gamma(\alpha)} {}_t\mathbf{D}^{\alpha}_{t_2-\tau} \int_{t_2-\tau}^{t_2} \left[{}_t\mathbf{D}^{\alpha}_{t_2}\left({}_{t_1}\mathbf{D}^{\alpha}_{t} y(t)\right)\right](z)(z-t)^{\alpha-1}\,\mathrm{d}z = 0, \tag{32}$$

for $t_1 \leqslant t \leqslant t_2 - \tau$ and

$${}_t\mathbf{D}^{\alpha}_{t_2}\left({}_{t_1}\mathbf{D}^{\alpha}_{t} y(t)\right) = 0, \tag{33}$$

for $t_2 - \tau \leqslant t \leqslant t_2$. We observe that when the delay is removed as well as $\alpha \to 1$ the classical Euler–Lagrange equations are reobtained.

5. Conclusion

The variational principles of mechanics are general, invariant, mathematically formulated statements from which we can deductively derive classical mechanics as a part of physical theory. The generalization of the classical variational principles leads us to some new type of Euler–Lagrange equations involving both the left and the right derivatives. These aspects are coming into the picture due to the fractional integration by parts. The corresponding equations are good candidates in describing the nonlocal effects. In previous studies in this area the potential part was kept local but the non-locality was added by replacing the classical derivatives with the fractional ones. However, new methods should be developed in the field of fractional calculus in order to describe better the complex systems possessing non-local effects.

In this paper we have obtained the Euler–Lagrange equations for a Lagrangian containing the fractional derivatives and the delay. The fractional integration by parts in the presence of the delay was obtained and used further to derive the corresponding Euler–Lagrange equations. When the delay is absent the corresponding fractional Euler–Lagrange equations are obtained. The classical Euler–Lagrange equations are reobtained when $\alpha \to 1$ and the delay is absent.

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